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ARE STATISTICAL LINEARIZATION AND STANDARD EQUIVALENT LINEARIZATION THE SAME METHODS IN THE ANALYSIS OF STOCHASTIC DYNAMIC SYSTEMS?

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1. INTRODUCTION

Linearization techniques are the most popular analytical tools in the determination of response characteristics of non-linear dynamic systems. Historically, the earliest work in the theory of statistical linearization was carried out simultaneously by Botton [1] and Kazakov [2]. The objective of this method is to replace the non-linear elements in a model by linear forms, where the coefficients of linearization can be found using the specific criterion of linearization. They used this approach to determine the characteristics of the responses of stochastic dynamic systems. Caughey [3–5] proposed the idea of replacing a non-linear oscillator with external Gaussian excitation by a linear one with the same excitation. He called this approach equivalent linearization, similar to Krilov and Bogoliubov [6], who studied deterministic vibrating systems by asymptotic methods. Since in the literature there are several methods called equivalent linearization, in this paper, we will call the Caughey approach presented in reference [5] standard equivalent linearization. The statistical linearization technique proposed by Kazakov [2] and the standard equivalent linearization are mainly treated in the literature as the same methods. However, some authors in their papers or books introduce different names for those techniques. For instance, in the book of Roberts and Spanos [7] statistical linearization in "Kazakov's sense" is described in the section entitled "Nonlinear elements without memory". Similarly, in the book of Soong and Grigoriu [8] statistical linearization is introduced in section 6.4.1, entitled "Memoryless Transformations" while equivalent linearization is introduced in section 6.4.2 entitled "Transformations with Memory". Also, in a survey paper by Socha and Soong [9] both approaches were separately reviewed. However, it is not mentioned in any paper or book if the statistical and equivalent linearization methods are the same, or the conditions under which they are realized or discussed. This problem was partially discussed in the book by Roberts and Spanos [7].

The objective of this paper is to re-derive both approaches with the mean-square criterion of the equivalency and Gaussian closure and to establish under what conditions both linearization techniques are the same. For clarity of presentation, the study is restricted to a non-linear oscillator excited by a stationary Gaussian white noise.

Consider a non-linear oscillator described by

$$\mathrm{d}\mathbf{X} = \mathbf{F}(\mathbf{X})\,\mathrm{d}t + \mathbf{G}\,\mathrm{d}\xi, \tag{1}$$

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where

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{F}(\mathbf{X}) = \begin{bmatrix} x_2 \\ -2h\omega_0 x_2 - \omega_0^2 x_1 - \psi(x_1, x_2) \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 0 \\ q \end{bmatrix}, \tag{2}$$

ω_0, h and q are constant parameters; $\psi(x_1, x_2)$ is a non-linear function such that $\psi(0, 0) = 0$ and $\xi(t)$ is a standard Wiener process. For simplicity, the study is restricted to a stationary case and it is assumed that sufficient conditions of the existence of the solution of equation (1) are satisfied. The application of a linearization technique gives the linearized oscillator in the form

$$\mathrm{d}\mathbf{X} = \mathbf{B}(k_1, k_2)\mathbf{X}\,\mathrm{d}t + \mathbf{G}\mathrm{d}\xi, \tag{3}$$

where

$$\mathbf{B}(k_1, k_2) = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}. \tag{4}$$

2. STATISTICAL LINEARIZATION

Two cases corresponding to two types of non-linearities are considered.

Case (a): In the Kazakov approach [2], the objective of statistical linearization is to find for the non-linear function

$$y = f_2(x_1, x_2) = \omega_0^2 x_1 + 2h\omega_0 x_2 + \psi(x_1, x_2), \tag{5}$$

a linear function

$$y = k_1 x_1 + k_2 x_2, \tag{6}$$

where k_1 and k_2 are linearization coefficients that minimizes the mean-square error defined by

$$I_1 = E[(f_2(x_1, x_2) - k_1 x_1 - k_2 x_2)^2], \tag{7}$$

where the averaging operation in equality (7) is defined by a non-Gaussian probability density function of a two-dimensional variable.

$$E_{n2}[\cdot] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [\cdot]\, g_{n2}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{8}$$

where $g_{n2}(x_1, x_2)$ is the joint probability density function of the two-dimensional input variable (x_1, x_2). Hence, the linearization coefficients k_1 and k_2 can be calculated from the following algebraic equations (necessary conditions of minimum of criterion (7)):

$$\begin{aligned} E_{n2}[x_1\psi(x_1, x_2)] - (\omega_0^2 + k_1)E_{n2}[x_1^2] - (2h\omega_0 + k_2)E_{n2}[x_1 x_2] = 0, \\ E_{n2}[x_2\psi(x_1, x_2)] - (\omega_0^2 + k_1)E_{n2}[x_1 x_2] - (2h\omega_0 + k_2)E_{n2}[x_2^2] = 0. \end{aligned} \tag{9}$$

Case (b): A particular case, when the non-linear function in equation (1) depends on one variable, for instance on x_1 is considered

$$y = f_1(x_1) = \omega_0^2 x_1 + \phi(x_1) \tag{10}$$

and a linearized function has the form

$$y = kx_1, \tag{11}$$

where k is a linearization coefficient. Then, a criterion of equivalency is taken as the mean-square error of displacement defined by

$$I_1 = E[(f_1(x_1) - kx_1)^2], \tag{12}$$

where the averaging operation in equality (12) is defined by a non-Gaussian probability density function of one-dimensional input variable $g_{n1}(x_1)$, i.e.,

$$E_{n1}[\cdot] = \int_{-\infty}^{+\infty} [\cdot]\, g_{n1}(x_1)\, \mathrm{d}x_1. \tag{13}$$

Hence, the linearization coefficient k has the following form:

$$k = \omega_0^2 + \frac{E_{n1}[x_1 \phi(x_1)]}{E_{n1}[x_1^2]}. \tag{14}$$

3. STANDARD EQUIVALENT LINEARIZATION

Case (a): In the Caughey approach [5], the objective of equivalent linearization is to find for the non-linear dynamic system (1) the equivalent linear dynamic system (3). Caughey suggested rewriting equation (1) in the form

$$\mathrm{d}\mathbf{X} = [\mathbf{KX} + \mathbf{F}(\mathbf{X}) - \mathbf{KX}]\, \mathrm{d}t + \mathbf{G}\, \mathrm{d}\xi, \tag{15}$$

where the matrix $\mathbf{K}$ has the form

$$\mathbf{K} = \mathbf{B}(k_1, k_2) = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \tag{16}$$

and to minimize the mean square criterion defined by

$$I_2 = E[(f_2(x_1, x_2) - k_1 x_1 - k_2 x_2)^2, \tag{17}$$

where the averaging operation in equality (17) is defined by a non-Gaussian probability density function of the two-dimensional variable $\mathbf{X} = [x_1, x_2]^{\mathrm{T}}$

$$E_{N2}[\cdot] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\cdot]\, g_{N2}(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2. \tag{18}$$

The probability density function $g_{N2}(x_1, x_2)$ can be found by solving the corresponding Fokker–Planck equation for system (1). Similar to the previous derivations, the linearization coefficients k_1 and k_2 can be calculated from the following algebraic equations (necessary conditions of minimum of criterion (17)):

$$\begin{aligned} &E_{N2}[x_1 \psi(x_1, x_2)] - (\omega_0^2 + k_1) E_{N2}[x_1^2] - (2h\omega_0 + k_2) E_{N2}[x_1 x_2] = 0, \\ &E_{N2}[x_2 \psi(x_1, x_2)] - (\omega_0^2 + k_1) E_{N2}[x_1 x_2] - (2h\omega_0 + k_2) E_{N2}[x_2^2] = 0. \end{aligned} \tag{19}$$

Case (b): Caughey in reference [5] also considered an example of a non-linear oscillator when the function ψ depends on one variable, i.e., $\psi(x_1, x_2) = \phi(x_1)$. As in the previous case, Caughey suggested rewriting equation (1) in form (15), where the matrix $\mathbf{K}$ has the form

$$\mathbf{K} = \mathbf{B}(k) = \begin{bmatrix} 0 & 1 \\ -k & -2h\omega_0 \end{bmatrix} \tag{20}$$

and to minimize the mean square error of displacements defined by

$$I_2 = E[f_1(x_1) - kx_1)^2], \tag{21}$$

where the averaging operation in equality (21) is defined by a non-Gaussian probability density function of the two-dimensional variable $\mathbf{X} = [x_1, x_2]^{\mathrm{T}}$.

$$E_{N1}[\cdot] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [\cdot] g_{N1}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{22}$$

where $g_{N1}(x_1, x_2)$ is the probability density function of two-dimensional response vector state (solution of equation (1)). It can be found by solving the corresponding Fokker–Planck equation for system (1) where the non-linearity $\psi(x_1, x_2)$ is replaced by $\phi(x_1)$. In this case, the solution can be found in an analytical form and is given by

$$g_{N1}(x_1, x_2) = \frac{1}{c_{N1}} \exp\left\{ -\frac{2h\omega_0}{q^2}\left(\omega_0^2 x_1^2 + \int_0^{x_1} \phi(s)\,\mathrm{d}s + x_2^2\right)\right\}, \tag{23}$$

where c_{N1} is a normalized constant.

In the particular case of the Duffing oscillator, it has the form

$$g_N(x_1, x_2) = \frac{1}{c_N} \exp\left\{ -\frac{2h\omega_0}{q^2}\left(\omega_0^2 x_1^2 + \varepsilon\frac{x_1^4}{2} + x_2^2\right)\right\}. \tag{24}$$

Hence, the linearization coefficient k has the form

$$k = \omega_0^2 + \frac{E_{N1}[x_1\phi(x_1)]}{E_{N1}[x_1^2]}. \tag{25}$$

It should be stressed, that many authors in their derivations of equivalent linearization do not define the averaging operation in equality (21). Some authors even made an error by assuming that the probability density function $g_{Ni}(x_1, x_2)$, $i = 1, 2$ is replaced by a Gaussian one corresponding to the solution of a linearized system. They carryout this replacement before differentiating with respect to the linearization coefficients. It has been discussed, for instance in references [10–13].

4. DETERMINATION OF RESPONSE CHARACTERISTICS

Following the remark from reference [7], it is noted that if for an isolated element the distribution of the input process is known, then the evaluation of the expected quantities in expression (9) or (14) is straightforward. However, when the non-linear element is incorporated into an overall system, the distribution of the input to the non-linear element is unknown. Therefore, in the general case, the probability density functions for non-linear system (1) $g_{ni}(x_1)$ and $g_{Ni}(x_1, x_2)$, $i = 1, 2$ cannot be determined analytically and both authors Kazakov [2] and Caughey [5] suggested replacing them by their Gaussian approximations.

Case (a) *for statistical linearization*: The probability density of the input two-dimensional Gaussian variable defined by equation (6) has the form

$$g_{l2}(x_1, x_2) = \frac{1}{2\pi\sqrt{|\mathbf{K}_G|}} \exp\left\{ -\frac{1}{2}[x_1 x_2]\,\mathbf{K}_G^{-1}\,[x_1 x_2]^{\mathrm{T}} \right\}, \tag{26}$$

where $\mathbf{K}_G$ is the covariance matrix of the input two-dimensional Gaussian variable

$$\mathbf{K}_G = \begin{bmatrix} E[x_1^2] & E[x_1 x_2] \\ E[x_1 x_2] & E[x_2^2] \end{bmatrix}, \tag{27}$$

while the probability density function of the stationary response of linearized system (3) is given by

$$g_{L2}(x_1, x_2) = \frac{1}{c_{L2}} \frac{2k_2\sqrt{k_1}}{q^2} \exp\left\{ -\frac{k_2}{q^2}(k_1 x_1^2 + x_2^2) \right\}, \tag{28}$$

where c_{L2} is a normalized constant. The corresponding averaging operations have the form

$$E_{l2}[\cdot] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [\cdot]\, g_{l2}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{29}$$

$$E_{L2}[\cdot] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [\cdot]\, g_{L2}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{30}$$

where $g_{l2}(x_1, x_2)$ and $g_{L2}(x_1, x_2)$ are defined by equations (26) and (28) respectively.

The stationary moment equations for linearized system (3) are described by

$$E_{L2}[x_1 x_2] = 0, \quad E_{L2}[x_2^2] - k_2 E_{L2}[x_1 x_2] - k_1 E_{L2}[x_1^2] = 0,$$
$$-2k_1 E_{L2}[x_1 x_2] - 2k_2 E_{L2}[x_2^2] = q^2. \tag{31}$$

To determine the response the characteristics an iterative procedure is used, where the moments in equalities (9) are replaced by the corresponding moments obtained by application of the averaging operation (30). These moments are solutions of equation (31) where Gaussian closure is applied. In case (a) of statistical linearization the iterative procedure has the form

Procedure 1a.

(1) Substitute $k_1 = \omega_0^2$ and $k_2 = 2h\omega_0$ and solve equations (31).
(2) Replace the moments in equalities (9) by corresponding solutions of (31) and determine new linearization coefficients k_1 and k_2 using Gaussian closure.
(3) Substitute the coefficients k_1 and k_2 into equations (31) and solve them.
(4) Repeat steps (2) and (3) until convergence.

Case (b) *for statistical linearization*: The probability density of the input one-dimensional Gaussian variable defined by equation (11) has the form

$$g_{l1}(x_1) = \frac{1}{\sqrt{2\pi}\sigma_{x_1}} \exp\left\{ -\frac{x_1^2}{2\sigma_{x_1}^2} \right\}, \tag{32}$$

where $\sigma_{x_1}^2 = E[x_1^2]$ is the variance of the input one-dimensional Gaussian variable, while the probability density function of the two-dimensional stationary response of linearized system (3) for $k_2 = 2h\omega_0$ and $k_1 = k$ is given by equation (28)

$$g_{L1}(x_1, x_2) = \frac{1}{c_{L1}} \frac{4h\omega_0\sqrt{k}}{q^2} \exp\left\{ -\frac{2h\omega_0}{q^2}(kx_1^2 + x_2^2)\right\}, \tag{33}$$

where c_{L1} is a normalized constant.

The corresponding averaging operations have the form

$$E_{l1}[\cdot] = \int_{-\infty}^{+\infty} [\cdot] g_{l1}(x_1)\,\mathrm{d}x_1, \tag{34}$$

$$E_{L1}[\cdot] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [\cdot] g_{L1}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{35}$$

where $g_{l1}(x_1)$ and $g_{L1}(x_1, x_2)$ are the probability density functions defined by equations (32) and (33) respectively. The stationary moments equations for linearized system (3) for $k_2 = 2h\omega_0$ and $k_1 = k$ are described by equation (31), where $E_{L2}[\cdot]$ is replaced by $E_{L1}[\cdot]$

$$E_{L1}[x_1x_2] = 0, \quad E_{L1}[x_2^2] - 2h\omega_0 E_{L1}[x_1x_2] - kE_{L1}[x_1^2] = 0,$$
$$-2kE_{L1}[x_1x_2] - 4h\omega_0 E_{L1}[x_2^2] = q^2. \tag{36}$$

To determine the response characteristics, a modified version of Procedure 1a is used.
Procedure 1b

(1) Substitute $k = \omega_0^2$ and solve equations (36).
(2) Replace the moments in equality (14) by corresponding solutions of equations (36) and determine new linearization coefficient k using Gaussian closure.
(3) Substitute coefficient k into equations (36) and solve them.
(4) Repeat steps (2) and (3) until convergence.

Also, in case (b) the moments in equality (14) are replaced by the corresponding moments obtained by the application of the averaging operation (35) and Gaussian closure. These moments are solutions of equations (36).

In the case of standard equivalent linearization, the same iterative procedures as for cases (a) and (b) of statistical linearization are used, where step (2) in Procedures 1a and 1b is replaced by the following one:

In case (a) of standard equivalent linearization

(2a′) Replace the moments in equalities (19) by corresponding solutions of equation (31) and determine new linearization coefficients k_1 and k_2 using Gaussian closure.

In case (b) of standard equivalent linearization

(2b′) Replace the moments in equality (25) by corresponding solutions of equations (36) and determine new linearization coefficient k using Gaussian closure.

The replacement of moments in the iterative procedures for equivalent linearization is equivalent to the replacement of the averaging operation (18) and (22) by operations (30) and (35) respectively. Since in equality (35), $[\cdot]$ is replaced by the function $\phi(x_1)$ the integration with respect to x_2 disappears (is independent of the integration with respect to x_1 and is equal to 1). Hence, it follows that the final results (stationary moments) obtained by statistical and standard equivalent linearization are the same.

5. CONCLUSIONS AND FINAL REMARKS

Two basic linearization techniques, statistical linearization and standard equivalent linearization, with mean-square criterion and Gaussian closure have been re-derived and application of both the techniques to the determination of stationary response characteristics of a non-linear oscillator has been studied. From the detailed discussion presented in the paper follows the main conclusion which is the answer for the question stated in the title. In the general case, statistical linearization and standard equivalent linearization are not the same methods. This statement follows directly from definitions of averaging operations in both cases (a) and (b). Although the mean-square criteria of equivalence and the necessary conditions of minimum of these criteria have the same algebraic structure for both linearization techniques, the definitions of the averaging operations appearing in these formulas are not the same. In case (a), the averaging operations given by equations (8) and (18) are defined by two different functions of two variables. The difference between the averaging operations is even greater in case (b) where the probability density function appearing in equation (13) is a function of one variable while the corresponding probability density function appearing in equation (22) is a function of two variables. From the presented iterative procedures, it follows that in all cases of statistical and standard equivalent linearization, the moments in equalities (9) or (14) and (19) or (25) are replaced by the corresponding moments obtained by the application of the averaging operation (34) or (35) respectively where Gaussian closure is applied. These moments are solutions of equations (31) or (36), respectively.

From the presented arguments, one can conclude that the differences between the considered linearization techniques in application to the determination of the response characteristics, are eliminated by iterative procedures. One can note that statistical linearization and standard equivalent linearization are exactly the same methods when in equation (15). The vector error $\boldsymbol{\varepsilon} = \mathbf{F}(\mathbf{X}) - \mathbf{KX}$ is treated as the difference between two static elements isolated from the considered dynamic system i.e. it is replaced by $\varepsilon = f_2(x_1, x_2) - k_1x_1 - k_2x_2$ or when the probability density functions $g_{n1}(x_1, x_2)$ and $g_{N1}(x_1, x_2)$ defined by equations (8) and (18), respectively, are assumed to be the same.

Finally, it should be noted that although the comparison of linearization techniques was presented for a non-linear oscillator it can also be done for non-linear multi-dimensional dynamic or static systems using results presented, for instance, in references [14] and [15].

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